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COMMENT

On exact solutions of the doubly anharmonic oscillator

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Abstract. It is shown that exact solutions of the doubly anharmonic oscillator in the form of integrals can exist if the oscillator is given a supersymmetric form and the coupling constants satisfy a supersymmetric constraint.

In recent years the doubly anharmonic oscillator problem has been studied quite extensively (Sobelman 1979, Damburg *et al* 1982, 1984, Chaudhury *et al* 1984). However, the eigenvalue equation for the doubly anharmonic oscillator is not generally exactly solvable and therefore exact solutions, whenever they exist, are of great interest. Exact solutions of this problem are of two types: solutions of the elementary type (Flessas 1979, Flessas and Das 1980, Khare 1981, Singh *et al* 1981) and solutions which are represented by integrals (Flessas 1981). In a previous paper (Roy and Roychoudhury 1987) we have shown that new solutions (as well as old ones) which are of elementary character can be obtained if the problem is cast in the framework of supersymmetric quantum mechanics (SUSYQM). Here our purpose is to show that integral solutions can also be obtained within the same framework and new solutions can also be obtained if the superpotential is suitably chosen.

We recall that in one dimension the Hamiltonian for a SUSYQM system is defined by (Cooper and Freedman 1983):

$$H^{ss} = \{Q^+, Q\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{1}$$

$$Q = (p + \mathrm{i}W) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2}$$

$$Q^+ = (p - \mathrm{i}W) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}$$

$$H_\pm = -\mathrm{d}^2/\mathrm{d}x^2 + V_\pm(x) \tag{4}$$

$$V_\pm(x) = W^2(x) \pm W'(x) \tag{5}$$

where Q and Q^+ are called the supercharges (generators of supersymmetry transformations) and $W(x)$ is the superpotential. An important property characterising a SUSYQM system is that if $|\Omega\rangle$ is a ground state it is annihilated by the supercharges, i.e.

$$Q|\Omega\rangle = Q^+|\Omega\rangle = 0. \tag{6}$$

Using explicit forms of Q and Q^{+} it is easily found that the ground-state wavefunctions are of the form

$$\begin{pmatrix} \varphi_{+}^{0}(x) \\ 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 \\ \varphi_{-}^{0}(x) \end{pmatrix} \tag{7}$$

where the functions $\varphi_{\pm}^{0}(x)$ are given by

$$\varphi_{\pm}^{0}(x) \sim \exp\left(\pm\int^{x} W(t)\,\mathrm{d}t\right). \tag{8}$$

However, the states given by (7) are not both physically acceptable ground states—the choice between the two ground states is dictated by the normalisability of the functions given by (8). The important point to note here is that if a physical ground state exists then it is unique and supersymmetry is unbroken and the ground-state energy is identically zero.

Next we have to determine for what choice of the superpotential we can have a doubly anharmonic oscillator. To this end, we make the following choice:

$$W(x) = ax^3 + bx + \frac{c}{x} + \sum_{i=0}^{N} \frac{2g_i x}{(1+g_i x^2)} \qquad g_0 = 0. \tag{9}$$

(We have chosen this superpotential because the summation term in it has the special feature of reproducing terms containing x^2 and the rest cancels when we form $V_+(x)$.)

Let us first consider the case $N = 0$. In this case the scalar potential corresponding to the bosonic $(-)$ and fermionic $(+)$ sectors can be obtained from (5) and are given by

$$V_-(x) = a^2x^6 + 2abx^4 + (b^2 + 2ac - 3a)x^2 + c(c+1)/x^2 + (2bc - b) \tag{10}$$

$$V_+(x) = a^2x^6 + 2abx^4 + (b^2 + 2ac + 3a)x^2 + c(c-1)/x^2 + (2bc + b). \tag{11}$$

Next we identify the Fermi sector, i.e. $V_+(x)$, with the doubly anharmonic oscillator potential

$$V(x) = \tfrac{1}{3}\eta x^6 + \tfrac{1}{2}\lambda x^4 + w^2x^2. \tag{12}$$

(Under appropriate changes of sign of the parameters one can also identify $V_-(x)$ with $V(x)$.) If $V_+(x)$ and $V(x)$ are identical then we have

$$a^2 = \tfrac{1}{3}\eta \tag{13}$$

$$2ab = \tfrac{1}{2}\lambda \tag{14}$$

$$b^2 + 2ac + 3a = w^2 \tag{15}$$

$$c = 0, 1 \tag{16}$$

and the relation between the energy eigenvalues is given by

$$E = E_+ - b(2c+1) \tag{17}$$

(where E is the energy corresponding to $V(x)$ and E_+ is the energy corresponding to $V_+(x)$). From (13)-(15) the supersymmetric constraint is found to be

$$w^2 = 3\lambda^2/16\eta + (2c+1)(\eta/3)^{1/2}. \tag{18}$$

We now consider the following Riccati equation:

$$W_1^2(x) + W_1'(x) = W^2(x) + W'(x). \tag{19}$$

A trivial solution of this equation is given by

$$W_1(x) = W(x) \tag{20}$$

and a non-trivial solution is given by (Roy and Roychoudhury 1986a)

$$W_1(x) = W(x) + u(x) \tag{21}$$

where

$$u(x) = \exp\left(-2\int^x W(t)\,\mathrm{d}t\right)\left[K + \int^x \exp\left(-2\int^y W(t)\,\mathrm{d}t\right)\mathrm{d}y\right]^{-1} \tag{22}$$

where K is a constant of integration.

Hence the zero-energy solution corresponding to (21) is given by

$$\varphi_+^0(x) \sim \exp\left(\int^x W_1(z)\,\mathrm{d}z\right) = \exp\left(\int^x (W(z) + u(z)\,\mathrm{d}z)\right). \tag{23}$$

Therefore for $c = 0$ we get

$$\varphi_+^0(x) \sim \exp[\tfrac{1}{4}(\eta/3)^{1/2}x^4/4 + \tfrac{1}{8}(3/\eta)^{1/2}x^2]$$
$$\times\left(K + \int^{x^2} z^{-3/2}\exp[-\tfrac{1}{2}(\eta/3)^{1/2}z^2 - \tfrac{1}{4}(3/\eta)^{1/2}z]\,\mathrm{d}z\right) \tag{24}$$

$$E = -\tfrac{1}{4}\lambda(3/\eta)^{1/2} \tag{25}$$

since $E_+ = 0$, provided

$$w^2 = 3\lambda^2/16\eta + (3/\eta)^{1/2}. \tag{26}$$

This result was previously obtained by Flessas (1981). However, before accepting this solution we note the following points. It is known that if SUSY is unbroken the ground state (of zero energy) is unique (Cooper and Freedman 1983, Roy and Roychoudhury 1986b). Now if $c = 0$, then $\varphi_-^0(x) = \exp(-\int^x W(t)\,\mathrm{d}t)$ is normalisable and therefore no other ground state can exist (since in that case the ground state would be degenerate, which is impossible) and $\varphi_+^0(x) = \exp(\int^x W(t)\,\mathrm{d}t)$ and $\varphi_+^0(x) = \exp(\int^x W_1(t)\,\mathrm{d}t)$ are therefore not normalisable (it can also be checked explicitly).

We now turn to the case $c = 1$. In this case neither of $\varphi_\pm^0(x) \sim \exp(\pm\int^x W(t)\,\mathrm{d}t)$ are normalisable. From (23) we then have

$$\varphi_+^0(x) \sim -\exp[-\tfrac{1}{4}(\eta/3)^{1/2}x^4 - \tfrac{1}{8}\lambda(3/\eta)^{1/2}x^2]$$
$$- x\exp[\tfrac{1}{4}(\eta/3)^{1/2}x^4 + \tfrac{1}{8}\lambda(3/\eta)^{1/2}x^2]$$
$$\times\int^{x^2} z^{-1/2}[(\eta/3)^{1/2}z + \tfrac{1}{4}\lambda(3/\eta)^{1/2}]$$
$$\times\exp[-\tfrac{1}{2}(\eta/3)^{1/2}z^2 - \tfrac{1}{4}\lambda(3/\eta)^{1/2}z]\,\mathrm{d}z \tag{27}$$

$$E = -\tfrac{3}{4}\lambda(3/\eta)^{1/2} \tag{28}$$

provided

$$w^2 = 3\lambda^2/16\eta + 5(\eta/2)^{1/2}. \tag{29}$$

The result (27) was also obtained by Flessas (1981) and we do not repeat the arguments to show that $\varphi_+^0(x)$ behaves properly at $x = \pm\infty$ and $x = 0$.

Next to obtain a new integral solution let us consider $N=1$ in (9). Then from (5) we have

$$V_+(x)=a^2x^6+2abx^4+(b^2+2ac+7a)x^2+c(c-1)/x^2+(2bc+5b-4a/g_1)$$
$$+(4a/g_1-4b+2g_1+4cg_1)/(1+g_1x^2). \tag{30}$$

As before, comparing (30) with (12) we obtain

$$a^2=\tfrac{1}{3}\eta \tag{31}$$
$$2ab=\tfrac{1}{2}\lambda \tag{32}$$
$$b^2+2ac+7a=w^2 \tag{33}$$
$$4a/g_1-4b+2g_1(2c+1)=0 \tag{34}$$
$$c=0,1 \tag{35}$$

and the relation between energy eigenvalues is

$$E=E_+-b(2c+5)+4a/g_1. \tag{36}$$

Now for identical reasons as before we have to reject the solution corresponding to $c=0$. Using (21) and (22) the solution corresponding to $c=1$ is found to be

$$\varphi_+^0(x)\sim x(1+g_1x^2)\exp[\tfrac{1}{4}(\eta/3)^{1/2}x^4+\tfrac{1}{8}\lambda(3/\eta)^{1/2}x^2]$$
$$\times\Bigg[-x^{-1}(1+g_1x^2)^{-2}\exp[-\tfrac{1}{2}(\eta/3)^{1/2}x^4-\tfrac{1}{4}\lambda(3/\eta)^{1/2}x^2]$$
$$-\int^{x^2}\mathrm{d}z\,z^{-1/2}\left(\frac{g_1}{(1+g_1z)^3}+\frac{(\eta/3)^{1/2}z+\frac{1}{4}\lambda(3/\eta)^{1/2}}{(1+g_1z)^2}\right)$$
$$\times\exp[-\tfrac{1}{2}(\eta/3)^{1/2}z-\tfrac{1}{4}\lambda(3/\eta)^{1/2}z]\Bigg] \tag{37}$$

$$E=\tfrac{5}{4}\lambda(3/\eta)^{1/2}\pm2[3\lambda^2/16\eta-(3\eta)^{1/2}]^{1/2} \tag{38}$$
$$g_1=\tfrac{1}{12}\lambda((3/\eta)^{1/2}\pm\tfrac{1}{3}[3\lambda^2/16\eta-(12\eta)^{1/2}]^{1/2} \tag{39}$$

provided

$$w^2=3\lambda^2/16\eta+(27\eta)^{1/2}. \tag{40}$$

We note that the above solution, i.e. $\varphi_+^0(x)$, behaves correctly if

$$2gI_3+\left(\frac{2(\eta/3)^{1/2}}{g_1}+\tfrac{1}{2}\lambda(3/\eta)^{1/2}\right)I_1=\frac{2(\eta/3)^{1/2}}{g_1}I_2 \tag{41}$$

where

$$I_n=\int_0^\infty\frac{\mathrm{d}z}{(1+g_1z^2)^n}\exp[-\tfrac{1}{2}(\eta/3)^{1/2}z^2-\tfrac{1}{4}\lambda(3/\eta)^{1/2}z]. \tag{42}$$

Equation (42) is a relation between the parameters λ, η and g and for certain values of the parameters this is satisfied. This equation should be compared with equation (14) of Flessas (1981). In this context also see the paper by Leaute and Marcilhacy (1986) who also obtained a transcendental equation involving the parameters in the case of non-polynomial solutions.

Now if we consider $N=2,3,\ldots$ and proceed as before we can obtain more integral solutions (although in these cases the equation analogous to (41) would be more complicated, the relations are neverthless exact and can always be tackled numerically to give exact solutions).

In conclusion we have shown that all the solutions of the doubly anharmonic oscillator can be obtained within a common framework and they all correspond to the ground state of SUSYQM systems.

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